THE NUMBER OF COMPOSITIONS INTO POWERS OF b

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Abstract. For a fixed integer base $b \geq 2$, we consider the number of compositions of 1 into a given number of powers of b and, related, the maximum number of representations a positive integer can have as an ordered sum of powers of b.

We study the asymptotic growth of those numbers and give precise asymptotic formulae for them, thereby improving on earlier results of Molteni. Our approach uses generating functions, which we obtain from infinite transfer matrices.

1. Introduction

Representations of integers as sums of powers of 2 occur in various contexts, most notably of course in the usual binary representation. *Partitions* of integers into powers of 2, i.e., representations of the form

$$\ell = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_n} \tag{1.1}$$

with nonnegative integers $a_1 \geq a_2 \geq \cdots \geq a_n$ (not necessarily distinct!) are also known as *Mahler partitions* (see [2, 11, 14, 18]).

The number of such partitions exhibits interesting periodic fluctuations. The situation changes, however, when *compositions* into powers of 2 are considered, i.e., when the summands are arranged in an order. In other words, we consider representations of the form (1.1) without further restrictions on the exponents a_1, a_2, …, a_n other than being nonnegative.

Motivated by the study of the exponential sum

$$s(\xi) = \sum_{r=1}^{\tau} \xi^{2^r},$$

where ξ is a primitive qth root of unity and τ is the order of 2 modulo q (see [15]), Molteni [16] recently studied the maximum number of representations a positive integer can have as an ordered sum of n powers of 2. More generally, fix an integer $b \geq 2$, let

$$\mathcal{U}_b(\ell, n) = \#\{(a_1, a_2, \dots, a_n) \in \mathbb{N}_0^n \mid b^{a_1} + b^{a_2} + \cdots + b^{a_n} = \ell\} \tag{1.2}$$

be the number of representations of ℓ as an ordered sum of n powers of b, and let $\mathcal{W}_b(s, n)$ be the maximum of $\mathcal{U}_b(\ell, n)$ over all positive integers ℓ with b-ary sum of digits equal to s. It was shown in [15] that

$$\frac{\mathcal{W}_2(s, n)}{n!} = \sum_{\substack{k_1, k_2, \dots, k_r \geq 1 \\ k_1 + k_2 + \cdots + k_r = n}} \prod_{j=1}^{r} \frac{\mathcal{W}_2(1, k_j)}{k_j!}, \tag{1.3}$$

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which generalizes in a straightforward fashion to arbitrary bases b. So knowledge of $\mathcal{W}_b(1,n)$ is the key to understanding $\mathcal{W}_b(s,n)$ for arbitrary s.

For the moment, let us consider the case $b=2$. There is an equivalent characterisation of $\mathcal{W}_2(1,n)$ in terms of compositions of 1. To this end, note that the number of representations of $2^h\ell$ as a sum of powers of 2 is the same as the number of representations of ℓ for all integers h if negative exponents are allowed as well (simply multiply/divide everything by 2^h). Therefore, $\mathcal{W}_2(1,n)$ is also the number of solutions to the Diophantine equation

$$2^{-k_1}+2^{-k_2}+\cdots+2^{-k_n}=1 \tag{1.4}$$

with nonnegative integers $k_1,k_2,\ldots,k_n$, i.e., the number of *compositions* of 1 into powers of 2. This sequence starts with

$$1,1,3,13,75,525,4347,41245,441675,5259885,68958747,\ldots$$

and is A007178 in the On-Line Encyclopedia of Integer Sequences [17].

The main goal of this paper is to determine precise asymptotics for the number of such binary compositions as $n\to\infty$. Lehr, Shallit and Tromp [13] encountered these compositions in their work on automatic sequences and gave a first bound, namely

$$\mathcal{W}_2(1,n)/n! \le K\cdot 1.8^n$$

for some constant K. It was mainly based on an asymptotic formula for the number of *partitions* of 1 into powers of 2, which was derived independently in different contexts, cf. [1, 7, 12] for example (or see the recent paper of Elsholtz, Heuberger and Prodinger [5] for a detailed survey). This bound was further improved by Molteni, who gave the inequalities

$$0.3316\cdot(1.1305)^n \le \mathcal{W}_2(1,n)/n! \le (1.71186)^{n-1}\cdot n^{-1.6}$$

in [15]. Giorgilli and Molteni [9] provided an efficient recursive formula for $\mathcal{W}_2(1,n)$ and used it to prove an intriguing congruence property. In his recent paper [16], Molteni succeeded in proving the following result, thus also disproving a conjecture of Knuth on the asymptotic behaviour of $\mathcal{W}_2(1,n)$.

Theorem I (Molteni [16])**.** *The limit*

$$\gamma=\lim_{n\to\infty}(\mathcal{W}_2(1,n)/n!)^{1/n}=1.192674341213466032221288982528755\ldots$$

exists.

Molteni's argument is quite sophisticated and involves the study of the spectral radii of certain matrices. The aim of this paper will be to present a different approach to the asymptotics of $\mathcal{W}_2(1,n)$ (and more generally, $\mathcal{W}_2(s,n)$) by means of generating functions that allows us to obtain more precise information. Our main theorem reads as follows.

Theorem II. *There exist constants $\alpha=0.2963720490\ldots$, $\gamma=1.1926743412\ldots$ (as in Theorem I) and $\kappa=2/(3\gamma)<1$ such that*

$$\frac{\mathcal{W}_2(1,n)}{n!}=\alpha\gamma^n(1+O(\kappa^n)).$$

More generally, for every fixed s, there exists a polynomial $P_s(n)$ with leading term $\alpha^s n^{s-1}/(s-1)!$ such that

$$\frac{\mathcal{W}_2(s,n)}{n!}=P_s(n)\,\gamma^n(1+O(\kappa^n)).$$

We also prove a more general result for arbitrary bases instead of 2. Consider the Diophantine equation

$$b^{-k_1} + b^{-k_2} + \cdots + b^{-k_n} = 1. \tag{1.5}$$

Multiplying by the common denominator and taking the equation modulo $b-1$, we see that there can only be solutions if $n \equiv 1 \bmod b-1$, i.e., $n = (b-1)m+1$ for some nonnegative integer m. We write $q_b(m)$ for the number of solutions (n-tuples of nonnegative integers satisfying (1.5)) in this case. Note that $q_b(m)$ is also the maximum number of representations of an arbitrary power of b as an ordered sum of $n = (b-1)m+1$ powers of b. We have the following general asymptotic formula.

Theorem III. *For every positive integer $b \geq 2$, there exist constants $\alpha = \alpha_b$, $\gamma = \gamma_b$ and $\kappa = \kappa_b < 1$ such that the number $q_b(m)$ of compositions of 1 into $n = (b-1)m+1$ powers of b, which is also the maximum number $\mathcal{W}_b(1,n)$ of representations of a power of b as an ordered sum of n powers of b, satisfies*

$$\frac{\mathcal{W}_b(1,n)}{n!} = \frac{q_b(m)}{n!} = \alpha\gamma^m(1 + O(\kappa^m)).$$

More generally, the maximum number $\mathcal{W}_b(s,n)$ of representations of a positive integer with b-ary sum of digits s as an ordered sum of $n = (b-1)m+s$ powers of b is asymptotically given by

$$\frac{\mathcal{W}_b(s,n)}{n!} = P_{b,s}(m)\,\gamma^m(1 + O(\kappa^m)),$$

where $P_{b,s}(m)$ is a polynomial with leading term $\alpha^s m^{s-1}/(s-1)!$.

The key idea is to equip every *partition* of 1 into powers of 2 (or generally b) with a weight that essentially gives the number of ways it can be permuted to a composition, and to apply the recursive approach that was used to count partitions of 1: if $p_2(n)$ denotes the number of such partitions into n summands, then the remarkable generating function identity

$$\sum_{n=1}^{\infty} p_2(n)x^n = \frac{\sum_{j=0}^{\infty}(-1)^j x^{2^j-1}\prod_{i=1}^{j}\frac{x^{2^i-1}}{1-x^{2^i-1}}}{\sum_{j=0}^{\infty}(-1)^j\prod_{i=1}^{j}\frac{x^{2^i-1}}{1-x^{2^i-1}}} \tag{1.6}$$

holds, and this can be generalised to arbitrary bases b, see the recent paper of Elsholtz, Heuberger and Prodinger [5]. In our case, we do not succeed to obtain a similarly explicit formula for the generating function, but we can write it as the quotient of two determinants of infinite matrices and infer analytic information from it. The paper is organised as follows: we first describe the combinatorial argument that yields the generating function, a priori only within the ring of formal power series. We then study the expression obtained for the generating function in more detail to show that it can actually be written as the quotient of two entire functions. The rest of the proof is a straightforward application of residue calculus (using the classical Flajolet–Odlyzko singularity analysis [6]).

Finally, we consider the maximum of $\mathcal{U}_b(\ell,n)$ over all ℓ, for which we write

$$M_b(n) = \max_{\ell \geq 1} \mathcal{U}_b(\ell,n) = \max_{s \geq 1} \mathcal{W}_b(s,n)\,.$$

This means that $M_b(n)$ is the maximum possible number of representations of a positive integer as a sum of exactly n powers of b. Equivalently, it is the largest coefficient in the power series expansion of

$$\left(x + x^b + x^{b^2} + \cdots\right)^n.$$

When $b = 2$, Molteni [16] obtained the following bounds for this quantity:

$$(1.75218)^n \ll M_2(n)\,/n! \le (1.75772)^n.$$

The gap between the two estimates is already very small; we improve this a little further by providing the precise constant of exponential growth.

Theorem IV. *For a certain constant* $\nu = 1.7521819\ldots$ *(defined precisely in Section 5), we have*

$$M_2(n)\,/n! \le \nu^n$$

for all $n \ge 1$*, and the constant is optimal: we have the more precise asymptotic formula*

$$M_2(n)\,/n! \sim \lambda n^{-1/2}\nu^n$$

with $\lambda = 0.2769343\ldots$*.*

Again, Theorem IV holds for arbitrary integer bases $b \ge 2$ for some constants $\nu = \nu_b$ and $\lambda = \lambda_b$ (it will be explained precisely how they are obtained). This is formulated as Theorem V in Section 5.

2. The recursive approach

For our purposes, it will be most convenient to work in the setting of compositions of 1, i.e., we are interested in the number $q_b(m)$ of (ordered) solutions to the Diophantine equation (1.5), where $n = (b-1)m+1$, as explained in the introduction. Our first goal is to derive a recursion for $q_b(m)$ and some related quantities, which leads to a system of functional equations for the associated generating functions.

Let $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ be a solution to the Diophantine equation (1.5) with $k_1 \ge k_2 \ge \cdots \ge k_n$. We will refer to such an n-tuple as a "partition" (although technically the k_i are only the exponents in a partition). We denote by $\mathsf{c}(\mathbf{k})$ the number of ways to turn it into a composition. If a_0 is the number of zeros, a_1 the number of ones, etc. in $\mathbf{k}$, then we clearly have

$$\mathsf{c}(\mathbf{k}) = \frac{n!}{\prod_{j\ge 0} a_j!}.$$

The *weight* of a partition $\mathbf{k}$, denoted by $\mathsf{w}(\mathbf{k})$, is now simply defined as

$$\mathsf{w}(\mathbf{k}) = \frac{1}{\prod_{j\ge 0} a_j!} = \frac{\mathsf{c}(\mathbf{k})}{n!}.$$

Now let

$$\mathcal{P}_m = \Big\{\mathbf{k} = (k_1, k_2, \ldots, k_n) \,\Big|\, n = (b-1)m+1,\ b^{-k_1} + b^{-k_2} + \cdots + b^{-k_n} = 1,\ k_1 \ge k_2 \ge \cdots \ge k_n\Big\}$$

be the set of all partitions of 1 with $n = (b-1)m+1$ terms and, likewise,

$$\mathcal{C}_m = \Big\{\mathbf{k} = (k_1, k_2, \ldots, k_n) \,\Big|\, n = (b-1)m+1,\ b^{-k_1} + b^{-k_2} + \cdots + b^{-k_n} = 1\Big\}$$

the set of compositions. We obtain the formula

$$q_b(m) = \#\mathcal{C}_m = \sum_{\mathbf{k}\in\mathcal{P}_m} \mathsf{c}(\mathbf{k}) = n! \sum_{\mathbf{k}\in\mathcal{P}_m} \mathsf{w}(\mathbf{k})$$

for their number.

Our next step involves an important observation that is also used to obtain (1.6). Consider an element $\mathbf{k}$ of $\mathcal{P}_m$, and let r be the number of times the greatest element k_1 occurs (i.e.,

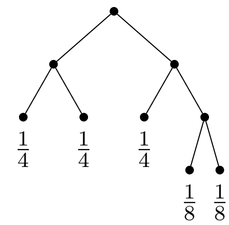

FIGURE 2.1. The canonical tree associated with the partition $1 = 3{\cdot}2^{-2}{+}2{\cdot}2^{-3}$ of 1 into powers of 2. This partition has weight $\frac{1}{12}$ and corresponds to 10 distinct compositions.

$k_1 = k_2 = \cdots = k_r > k_{r+1}$). This number must be divisible by b (as can be seen by multiplying (1.5) by b^{k_n}) unless $\mathbf{k}$ is the trivial partition, so we can replace them by r/b fractions with denominator b^{k_n-1}.

This process can be reversed. Given a partition $\mathbf{k}$ in which the largest element occurs r times, we can replace s, $1 \le s \le r$, of these fractions by bs fractions with denominator b^{k_n+1}. This recursive construction can be illustrated nicely by a tree structure, as in Figure 2.1 in the case $b = 2$. Each partition corresponds to a so-called canonical tree (see [5]), and vice versa. Note that if $\mathbf{k} \in P_m$, then the resulting partition $\mathbf{k}'$ lies in P_{m+s}, and we clearly have

$$\mathsf{w}(\mathbf{k}') = \mathsf{w}(\mathbf{k}) \cdot \frac{r!}{(r-s)!\,(bs)!}. \tag{2.1}$$

Now we can turn to generating functions. Let $\mathcal{P}_{m,r}$ be the subset of $\mathcal{P}_m$ that only contains partitions for which $k_1 = k_2 = \cdots = k_r > k_{r+1}$ (i.e., in (1.5), the largest exponent occurs exactly r times), and let $\mathcal{C}_{m,r}$ be the set of compositions obtained by permuting the terms of an element of $\mathcal{P}_{m,r}$. We define a generating function by

$$Q_r(x) = \sum_{m\ge 0} \frac{\#\mathcal{C}_{m,r}}{((b-1)m+1)!} x^m = \sum_{m\ge 0} \sum_{\mathbf{k}\in\mathcal{P}_{m,r}} \frac{\mathsf{c}(\mathbf{k})}{((b-1)m+1)!} x^m = \sum_{m\ge 0} \sum_{\mathbf{k}\in\mathcal{P}_{m,r}} \mathsf{w}(\mathbf{k}) x^m.$$

In view of the recursive relation described above and in particular (2.1), we have $Q_1(x) = 1$ and $Q_r(x) = 0$ for all other r not divisible by b, and for all $s \ge 1$ we obtain

$$\begin{aligned} Q_{bs}(x) &= \sum_{m\ge 0} \sum_{\mathbf{k}\in\mathcal{P}_{m,bs}} \mathsf{w}(\mathbf{k}) x^m = \sum_{r\ge s} \sum_{m\ge s} \sum_{\mathbf{k}\in\mathcal{P}_{m-s,r}} \mathsf{w}(\mathbf{k}) \frac{r!}{(r-s)!\,(bs)!} x^m \\ &= x^s \sum_{r\ge s} \frac{r!}{(r-s)!\,(bs)!} \sum_{m\ge s} \sum_{\mathbf{k}\in\mathcal{P}_{m-s,r}} \mathsf{w}(\mathbf{k}) x^{m-s} = x^s \sum_{r\ge s} \frac{r!}{(r-s)!\,(bs)!} \, Q_r(x)\,. \end{aligned} \tag{2.2}$$

This can be seen as an infinite system of linear equations. Define the infinite (column-)vector $\mathbf{V}(x) = (Q_b(x)\,, Q_{2b}(x)\,, Q_{3b}(x)\,, \ldots)^T$, and the infinite matrix $\mathbf{M}(x)$ by its entries

$$m_{ij} = \begin{cases} \frac{(bj)!\,x^i}{(bj-i)!\,(bi)!} & \text{if } i \le bj, \\ 0 & \text{otherwise.} \end{cases}$$

Then the identity (2.2) above turns into the matrix identity

$$\mathbf{V}(x) = \mathbf{M}(x)\mathbf{V}(x) + \frac{x}{b!}\mathbf{e}_1, \tag{2.3}$$

where $\mathbf{e}_1 = (1, 0, 0, \dots)^T$ denotes the first unit vector. Within the ring of formal power series, this readily yields

$$\mathbf{V}(x) = \frac{x}{b!}(\mathbf{I} - \mathbf{M}(x))^{-1}\mathbf{e}_1, \tag{2.4}$$

and the generating function

$$Q(x) = \sum_{r\geq 1} Q_r(x) = \sum_{m\geq 0} \frac{q_b(m)}{((b-1)m+1)!} x^m$$

(recall that $q_b(m)$ is the number of compositions of 1 into $n = (b-1)m + 1$ powers of b) is given by

$$Q(x) = 1 + \mathbf{1}^T\mathbf{V}(x) = 1 + \frac{x}{b!}\mathbf{1}^T(\mathbf{I} - \mathbf{M}(x))^{-1}\mathbf{e}_1.$$

For our asymptotic result, we will need the dominant singularity of $Q(x)$, i.e., the zero of $\det(\mathbf{I} - \mathbf{M}(x))$ that is closest to 0. It is not even completely obvious that this determinant is well-defined, but the reasoning is similar to a number of comparable problems.

As mentioned earlier, the determinant $T(x) = \det(\mathbf{I} - \mathbf{M}(x))$ exists a priori within the ring of formal power series, as the limit of the principal minor determinants. We can write it as

$$\det(\mathbf{I} - \mathbf{M}(x)) = \sum_{h\geq 0}(-1)^h \sum_{\substack{1\leq i_1<i_2<\cdots<i_h \\ i_1,\dots,i_h\in\mathbb{N}}} x^{i_1+i_2+\cdots+i_h} \sum_{\sigma}(\operatorname{sgn}\sigma) \prod_{k=1}^{h} \frac{(b\,\sigma(i_k))!}{(b\,\sigma(i_k) - i_k)!\,(bi_k)!}, \tag{2.5}$$

where the inner sum is over all permutations of $\{i_1, i_2, \dots, i_h\}$. Using Eaves' sufficient condition, cf. [4], we get at least convergence for $|x| < 1$.

Using the bounds in the appendix, we can show that the formal power series given by (2.5) defines an entire function. Write $n = i_1 + i_2 + \cdots + i_h$ for the exponent of x, and note that

$$\prod_{k=1}^{h} \frac{(b\,\sigma(i_k))!}{(bi_k)!} = 1,$$

which is independent of σ. We also have $\sum_{k=1}^{h}(b\,\sigma(i_k) - i_k) = (b-1)n$. This, combined with the inequality $a! \geq \exp(a(\log a - 1))$ and the fact that $f(x) = x(\log x - 1)$ is a convex function, gives us

$$\prod_{k=1}^{h}(b\,\sigma(i_k) - i_k)! \geq \exp\left(\frac{b-1}{2} n\log n + O(n)\right).$$

On the other hand, the number of choices for i_1, i_2, …, i_h given n is equal to the number of partitions $q(n)$ of n into distinct parts, which is well known to be $\exp\big(\pi\sqrt{n/3} + O(\log n)\big)$. Finally, since $h \leq \sqrt{2n} + 1$ because i_1, i_2, …, i_h are all distinct, Stirling's formula tells us that the number of permutations σ of i_1, i_2, …, i_h is at most $\exp\big(\sqrt{n/2}\log n + O(\sqrt{n})\big)$. It follows that the coefficient of x^n in (2.5) is at most

$$\exp\left(-\frac{b-1}{2} n\log n + O(n)\right).$$

Since this bound decays superexponentially, the determinant $T(x) = \det(\mathbf{I} - \mathbf{M}(x))$ is an entire function. The same is true (by the same argument) for

$$S(x) = \mathbf{1}^T \operatorname{adj}(\mathbf{I} - \mathbf{M}(x))\mathbf{e}_1 = \det(\mathbf{M}^*(x)),$$

where $\mathbf{M}^*$ is obtained from $\mathbf{I} - \mathbf{M}(x)$ by replacing the first row by $\mathbf{1}$. Hence we can write the generating function $Q(x)$ as

$$Q(x) = 1 + \frac{x}{b!}\frac{S(x)}{T(x)}, \tag{2.6}$$

where $S(x)$ and $T(x)$ are both entire functions. The singularities of $Q(x)$ are thus all poles, and it remains to determine the dominant singularity, i.e., the zero of $T(x) = \det(\mathbf{I} - \mathbf{M}(x))$ with smallest modulus.

3. Analysing the generating function

Infinite systems of functional equations appear quite frequently in the analysis of combinatorial problems, see for example the recent work of Drmota, Gittenberger and Morgenbesser [3]. Alas, their very general theorems are not applicable to our situation as the infinite matrix $\mathbf{M}$ does not represent an ℓ_p-operator (one of their main requirements), due to the fact that its entries increase (and tend to ∞) along rows. However, we can adapt some of their ideas to our setting.

The main result of this section is the following lemma, whose proof we only sketch.

Lemma 3.1. *For every $b \geq 2$, the generating function $Q(x)$ has a simple pole at a positive real point ρ_b and no other poles with modulus $< \rho_b + \epsilon_b$ for some $\epsilon_b > 0$.*

Sketch of Proof. By considering compositions of 1 consisting of $b-1$ copies of b^{-1}, b^{-2}, ..., b^{1-m} and b copies of b^{-m}, we see that

$$q_b(m) \geq \frac{((b-1)m+1)!}{((b-1)!)^{m-1}b!},$$

which allows us to conclude that the radius of convergence of $Q(x)$ is at most $(b-1)!$. Since all coefficients are positive, Pringsheim's theorem guarantees that the radius of convergence, which we denote by ρ_b, is also a singularity (a pole since $Q(x)$ is meromorphic).

Next we consider $w_r = \lim_{x \to \rho_b^-} (x - \rho_b)^p \, Q_r(x)$, where p is the pole order of ρ_b. Multiplying the matrix equation (2.3) by $(x - \rho_b)^p$ and taking the limit, we see that $\mathbf{w} = (w_1, w_2, \ldots)^T$ is a right eigenvector of $\mathbf{M}(\rho_b)$, and because this matrix has positive entries on and above the main diagonal, all w_r are positive. It follows that all functions $Q_r(x)$ have the same pole order (as $Q(x)$).

Now we split the identity (2.3) appropriately to obtain an equation for $Q_1(x)$ only, whose solution takes the form

$$Q_1(x) = \frac{x}{b!}\,(1 - R(x))^{-1}$$

for some function $R(x)$ which has only positive coefficients. Thus, $R(x) = 1$ has a unique positive real solution, which must be ρ_b (and is a simple zero). Moreover, by the triangle inequality there are no complex solutions of $R(x) = 1$ with the same modulus, which means that there are no further singularities of $Q_1(x)$ (and thus $Q(x)$) in a circle of radius $\rho_b + \epsilon_b$ around 0 for suitable $\epsilon_b > 0$. □

4. Getting the Asymptotics

In this section, we prove Theorems II and III, which give us constants α_b, γ_b and $\kappa_b < 1$ such that for $n = (b-1)m + 1$

$$\frac{\mathcal{W}_b(s,n)}{n!} = P_{b,s}(m)\,\gamma_b^m (1 + O(\kappa_b^m))$$

b	α	γ
2	0.296372	1.19268
3	0.279852	0.534502
4	0.236824	0.170268
5	0.196844	0.0419317
6	0.165917	0.00834837
7	0.142679	0.00138959
8	0.1249575	0.000198440

TABLE 4.1. Values (numerical approximations) for the constants of Theorem III.

holds, where $P_{b,s}(m)$ is a polynomial with leading term $\alpha_b^s m^{s-1}/(s-1)!$. Numerical values of the α_b and γ_b can be found in Table 4.1.

For easier reading, we skip the index b again, i.e., we set $\alpha = \alpha_b$, $\gamma = \gamma_b$ and so on. The proof is the same for all b, except for the fact that different constants occur. Thus, we restrict ourselves here to $b = 2$, where we also give explicit estimates for all the constants involved.

Proof of Theorem II. By now, we know that the function $Q(x)$ can be written as the quotient of two entire functions, cf. Lemma A.1. More specifically,

$$Q(x) = 1 + \frac{xS(x)}{2T(x)},$$

and the first few terms of these power series are given by

$$S(x) = \mathbf{1}^T \operatorname{adj}(I - \mathbf{M}(x))\mathbf{e}_1 = \det(M^*(x)) = 1 - \tfrac{5}{12}x^2 - \tfrac{1}{6}x^3 - \tfrac{1}{24}x^4 + \tfrac{1}{45}x^5 + \cdots$$

and

$$T(x) = \det(I - \mathbf{M}(x)) = 1 - x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{8}x^4 + \tfrac{3}{40}x^5 + \cdots .$$

As Lemma C.1 shows, $T(x)$ has exactly one zero x_0 (which is a simple zero) inside a disk of radius $\frac{3}{2}$ around 0. This is then a simple pole of $Q(x)$, as one checks that $S(x_0) \neq 0$. Thus we can directly apply singularity analysis [6] in the meromorphic setting (cf. Theorem IV.10 of [8]) to obtain

$$\frac{q_2(n)}{(n+1)!} = [x^n]Q(x) = -\frac{S(x_0)}{2\,T'(x_0)}x_0^{-n} + O((2/3)^n)\,.$$

In the general case (arbitrary s), we use the relation

$$\sum_{n=1}^{\infty} \frac{\mathcal{W}_b(s,n)}{n!}x^n = \Big(\sum_{n=1}^{\infty} \frac{\mathcal{W}_b(1,n)}{n!}x^n\Big)^s,$$

which follows from Equation (1.3). Once again, we make use of the fact here that the (exponential) generating function is meromorphic, cf. Section 2. The singular expansion at $x = 1/\gamma$ is given by

$$\sum_{n=1}^{\infty} \frac{\mathcal{W}_b(s,n)}{n!}x^n = \Big(\frac{\alpha}{1-\gamma x} + O(1)\Big)^s,$$

which has $\alpha^s/(1-\gamma x)^s$ as a main term. Once again, singularity analysis [6] yields the desired asymptotic formula with main term as indicated in the statement of the theorem. □

We close this section with a remark concerning numerical calculations.

Remark 4.1. To obtain numerical values of all the constants involved in the statement of our theorems, we use the bounds in Appendix A together with interval arithmetic.

Denote the polynomials consisting of the first N terms of $S(x)$ and $T(x)$, by $S_N(x)$ and $T_N(x)$, respectively. By the methods of Lemmas A.1 and A.3 and Remarks A.2 and A.4, which can all be found in the appendix, we get, for instance, that $|T'(x) - T'_{60}(x)| \le B_{T'_{60}}$ with $B_{T'_{60}} = 8.397 \cdot 10^{-12}$. We also have $|S(x) - S_{60}(x)| \le B_{S_{60}}$ with $B_{S_{60}} = 1.848 \cdot 10^{-13}$ for the function in the numerator of $Q(x)$. We plug x_0 into the approximations S_{60} and T'_{60} and use these bounds to obtain precise values (with guaranteed error estimates) for all the constants that occur in our formula.

If one does not insist on such explicit error bounds for the numerical approximations, one can get more precise results. Here, specifically, the first three terms in the asymptotic expansion are as follows (although the numerical approximations lack the "certifiability" of e.g. those in Table 4.1):

$$\begin{aligned}\mathcal{W}_2(1,n)\,/n! &= 0.29637204905352907558864864213 3\cdot 1.19267434121346603222128898252 9^n \\ &\quad + 0.11973633538363165349506855424 5\cdot 0.64342741814950007012057031850 9^n \\ &\quad + 0.01747836352103880070513843818 33\cdot (-0.51839777389933777286272735707 10)^n \\ &\quad + \cdots .\end{aligned}$$

5. Maximum Number of Representations

Let $\mathcal{U}_b(\ell, n)$ and $\mathcal{W}_b(s, n)$ be as defined in (1.2) in the introduction. In this section we analyze the function $M(n) = M_b(n)$, which equals the maximum of $\mathcal{U}_b(\ell, n)$ over all ℓ, i.e., we have

$$M(n) = \max_{\ell \ge 1} \mathcal{U}_b(\ell, n) = \max_{s \ge 1} \mathcal{W}_b(s, n)\,.$$

This gives the maximum number of representations any positive integer can have as the sum of exactly n powers of b. We get the following result.

Theorem V. *Let $W(x)$ be the generating function*

$$W(x) = \sum_{n=1}^{\infty} \frac{\mathcal{W}_b(1, n)}{n!} x^n.$$

Let θ be the unique positive real solution of the equation $W(\theta) = 1$, and set $\nu = 1/\theta$. Then we have

$$M(n)\,/n! \le \nu^n \tag{5.1}$$

for all $n \ge 1$, and the constant is optimal: We have the more precise asymptotic formula

$$M(n)\,/n! = \lambda n^{-1/2} \nu^n \bigl(1 + O\bigl(n^{-1/2}\bigr)\bigr)$$

with $\lambda = (b-1)\,\bigl(\theta\, W'(\theta)\,\sigma\sqrt{2\pi}\bigr)^{-1}$, where $\sigma > 0$ is defined by

$$\sigma^2 = \frac{W''(\theta)}{\theta\, W'(\theta)^3} - \frac{1}{\theta\, W'(\theta)} + \frac{1}{\theta^2\, W'(\theta)^2}.$$

Moreover, the maximum $M(n) = \max_{s \ge 1} \mathcal{W}_b(s, n)$ is attained at $s = \mu n + O(1)$ with the constant $\mu = (\theta\, W'(\theta))^{-1}$.

b	λ	θ	$\nu = 1/\theta$	μ	σ^2
2	0.27693430	0.57071698	1.75218196	0.44867215	0.41775807
3	0.70656285	0.84340237	1.18567368	0.66924459	0.57114748
4	1.70314663	0.95872521	1.04305174	0.87318716	0.37650717
5	4.20099030	0.99167231	1.00839763	0.96645454	0.13477198
6	10.61691472	0.99861115	1.00139078	0.99304650	0.03480989
7	28.28286119	0.99980159	1.00019845	0.99880929	0.00714564
8	80.09108610	0.99997520	1.00002480	0.99982638	0.00121534

TABLE 5.1. Values (numerical approximations) for the constants of Theorem V.

In Table 5.1, we are listing numerical values for the constants of Theorem V.

We start with the upper bound (5.1) of Theorem V, which is easy to obtain. Recall that Equation (1.3) gives us

$$\sum_{n=1}^{\infty} \frac{\mathcal{W}_b(s,n)}{n!} x^n = \left(\sum_{n=1}^{\infty} \frac{\mathcal{W}_b(1,n)}{n!} x^n \right)^s = W(x)^s .$$

Since $\theta > 0$ was chosen such that $W(\theta) = 1$, it clearly follows that

$$\sum_{n=1}^{\infty} \frac{\mathcal{W}_b(s,n)}{n!} \theta^n = 1,$$

hence $\mathcal{W}_b(s,n)\,/n! \le \theta^{-n}$ for all s and n, and taking the maximum over all $s \ge 1$ yields

$$M(n)\,/n! = \max_{s\ge 1} \mathcal{W}_b(s,n)\,/n! \le \theta^{-n} = \nu^n,$$

which is what we wanted to show. It remains to prove the asymptotic formula for $M(n)$. To this end, we consider the bivariate generating function

$$G(x,u) = 1 + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} \frac{\mathcal{W}_b(s,n)}{n!} x^n u^s = \sum_{s=0}^{\infty} W(x)^s\, u^s = \frac{1}{1 - u\,W(x)}.$$

In order to get $\max_{s\ge 1} \mathcal{W}_b(s,n)$, we show that the coefficients varying with s fulfil a local limit law (as n tends to ∞). The maximum is then attained at its mean.

Proof of Theorem V (Sketch). Set

$$g_n(u) = \sum_{s=1}^{\infty} \frac{\mathcal{W}_b(s,n)}{n!} u^s.$$

We extract g_n from the bivariate generating function $G(x,u)$, and we would like to determine its largest coefficient. Note that the coefficient of u^k in $g_n(u)$ can only be nonzero if $k \equiv n \mod (b-1)$. Now we proceed as in Theorem IX.9 (singularity perturbation for meromorphic functions) of Flajolet and Sedgewick [8]. Note that by Lemma D.2 in the appendix the function $G(x,1)$ has a dominant simple pole at $x = \theta$. There exists a nonconstant function $\theta(u)$ with the following properties: it is analytic at $u = 1$, it fulfils $\theta(1) = \theta$, and we have $W(\theta(u)) = 1/u$. Moreover, for some $\epsilon > 0$ and u in a suitable neighbourhood of 1, there is no $x \ne \theta(u)$ with $W(x) = 1/u$ and $|x| \le \theta + \epsilon$. Finally, $\left|\theta(e^{i\phi})\right|$ attains its minimum at $\phi = 2a\pi/(b-1)$ ($a \in \{0, 1, \dots, b-2\}$) and nowhere else.

By Cauchy's integral formula and the residue theorem, we obtain

$$g_n(u) = \frac{1}{u\,\theta(u)\,W'(\theta(u))}\left(\frac{1}{\theta(u)}\right)^n + O\big((\theta+\epsilon)^{-n}\big)$$

for u in a suitable neighbourhood of 1.

The get the statement of Theorem V, we use the local version of the quasi-power theorem, see Theorem IX.14 of [8] or Hwang's original paper [10], in a suitably modified version to account for the fact that only every $(b-1)$-th coefficient is nonzero. □

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Appendix A. Bounds and Numerical calculations

In this section of the appendix the two formal power series

$$T(x) = \sum_{n\geq 0} t_n x^n = \det(I - \mathbf{M}(x))$$

and

$$S(x) = \sum_{n\geq 0} s_n x^n = \mathbf{1}^T \operatorname{adj}(I - \mathbf{M}(x))\mathbf{e}_1$$

of Section 2 (in particular cf. Equations (2.5) and (2.6)) are analyzed. Other (similar) functions arising on the way can be dealt with in a similar fashion.

Note that $S(x)$ is the determinant of a matrix, which is obtained by replacing the first row of $I - \mathbf{M}(x)$ by $\mathbf{1}$.

We find bounds for the coefficients t_n and s_n, which will be needed for numerical calculations with guaranteed error estimates. Further, those bounds will tell us that the two functions $T(x)$ and $S(x)$ are entire.

Lemma A.1. *The coefficient t_n satisfies the bound*

$$|t_n| \leq \exp\left(-\frac{b-1}{2} n \log n - cn + n\, g(n)\right)$$

with $c = (b-1)\left(\log \frac{b-1}{2\sqrt{2}} - 1\right)$ and with a decreasing function $g(n)$, which tends to zero as $n \to \infty$. In particular, the formal power series T defines an entire function. The same is true for the formal power series S. More precisely, we have

$$|s_n| \leq |t_n| + b!\,|t_{n+1}|\,.$$

Therefore, the coefficient s_n satisfies the bound

$$|s_n| \leq (b!+1) \exp\left(-\frac{b-1}{2} n \log n - cn + n\, g(n)\right).$$

Proof. Recall expression (2.5) for the determinant:

$$\det(\mathbf{I} - \mathbf{M}(x)) = \sum_{h\geq 0} (-1)^h \sum_{\substack{1\leq i_1<i_2<\cdots<i_h \\ i_1,\ldots,i_h\in\mathbb{N}}} x^{i_1+i_2+\cdots+i_h} \sum_{\sigma} (\operatorname{sgn}\sigma) \prod_{k=1}^{h} \frac{(b\,\sigma(i_k))!}{(b\,\sigma(i_k) - i_k)!\,(bi_k)!}.$$

Write $n = i_1 + i_2 + \cdots + i_h$ for the exponent of x, and note that

$$\prod_{k=1}^{h} \frac{(b\,\sigma(i_k))!}{(bi_k)!} = 1,$$

which is independent of σ. Now we have

$$\sum_{k=1}^{h} (b\,\sigma(i_k) - i_k) = (b-1) \sum_{k=1}^{h} i_k = (b-1)n.$$

Since $a! \geq \exp(a(\log a - 1))$ for all positive integers a and $f(x) = x(\log x - 1)$ is a convex function, we have

$$\begin{aligned}\prod_{k=1}^{h} (b\,\sigma(i_k) - i_k)! &\geq \exp\left(\sum_{k=1}^{h} (b\,\sigma(i_k) - i_k)\left(\log(b\,\sigma(i_k) - i_k) - 1\right)\right) \\ &\geq \exp\left(h\frac{(b-1)n}{h}\left(\log\frac{(b-1)n}{h} - 1\right)\right) \\ &= \exp\left((b-1)n\left(\log\frac{(b-1)n}{h} - 1\right)\right).\end{aligned}$$

Since $i_1, i_2, \dots, i_h$ have to be distinct, we also have

$$n = i_1 + i_2 + \cdots + i_h \geq 1 + 2 + \cdots + h = \frac{h(h-1)}{2}.$$

Thus $h \leq \sqrt{2n} + 1$, which means that

$$\prod_{k=1}^{h} (b\,\sigma(i_k) - i_k)! \geq \exp\left(\frac{b-1}{2} n \log n + (b-1)n\left(\log\frac{b-1}{2\sqrt{2}} - 1\right)\right).$$

Now that we have an estimate for each term in (2.5), let us also determine a bound for the number of terms corresponding to each exponent n.

It is well known that the number of partitions $q(n)$ of n into distinct parts is asymptotically equal to $\exp\left(\pi\sqrt{n/3} + O(\log n)\right)$. In Robbins's paper [19] we can find the upper bound[1]

$$q(n) \leq \frac{\pi}{\sqrt{12n}} \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n} + \frac{\pi^2}{12}\right).$$

For each choice of $\{i_1, i_2, \dots, i_h\}$, there are at most $h!$ permutations σ, which can be bounded by Stirling's formula by

$$h! \leq \exp\left(h\log h - h + \frac{1}{2}\log h + 1\right) \leq \exp\left(\sqrt{\frac{n}{2}}\log 8n + \frac{3}{4}\log n + \frac{9}{4}\log 2\right).$$

It follows that the coefficient t_n of T is bounded (in absolute values) by

$$\frac{\exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n} + \frac{\pi^2}{12} + \log\pi - \frac{1}{2}\log(12n) + \sqrt{\frac{n}{2}}\log 8n + \frac{3}{4}\log n + \frac{9}{4}\log 2\right)}{\exp\left(\frac{b-1}{2}n\log n + (b-1)n\left(\log\frac{b-1}{2\sqrt{2}} - 1\right)\right)} \leq \exp\left(-\frac{b-1}{2}n\log n - cn + n\,g(n)\right),$$

where $g(n)$ was defined suitably. Since this bound decays superexponentially, the determinant T is an entire function.

The same argument works for S. There, we split up into the summands where we have $i_1 = 1$ and all other summands. The second part (the summands with $i_1 > 1$) can be bounded

[1] Note that in the published version of Robbins [19] a constant in the main theorem is printed wrongly.

by the absolute value of the n-th coefficient of $\det(I - \mathbf{M}(x))$. Each of the summands with $i_1 = 1$ equals a summand of $\det(I - \mathbf{M}(x))$ multiplied by the factor

$$-\frac{(b\,\sigma(i_1) - i_1)!\,(bi_1)!}{(b\,\sigma(i_1))!\,x} = -\frac{b!}{x}\frac{(b\,\sigma(1) - 1)!}{(b\,\sigma(1))!} = -\frac{(b-1)!}{x\,\sigma(1)}$$

or is zero (when $\sigma(i_1) = 1$). Therefore, the summands occurring for the n-th coefficient of the determinant S can be bounded by the absolute value of the coefficient of x^{n+1} of $\det(I - \mathbf{M}(x))$ times $b!$. This leads the desired result. The "therefore"-part follows, since $g(n)$ is decreasing, which can be checked easily. □

Remark A.2. The bounds of Lemma A.1 for the determinant (2.5) can be tightened: Instead of the constant $c = (b-1)\left(\log\frac{b-1}{2\sqrt{2}} - 1\right)$ we can use $(b-1)\left(\log\frac{b-1}{\sqrt{2}+1/\sqrt{n}} - 1\right)$. For an explicit n, we can calculate $g(n)$ more precisely by using the number of partitions of n into distinct parts (and not a bound for that number) and similarly by using the factorial directly instead of Stirling's formula.

An even better, but less explicit bound for the n-th coefficient of $\det(I - \mathbf{M}(x))$ is given by

$$|t_n| \le \sum_{h\ge0} h! \sum_{\substack{1\le i_1<i_2<\cdots<i_h\\ i_1,\ldots,i_h\in\mathbb{N}\\ i_1+i_2+\cdots+i_h=n}} \exp\left(-(b-1)n\left(\log\frac{(b-1)n}{h} - 1\right)\right). \tag{A.1}$$

Note that we do not know whether this bound is decreasing in n or not. However, for a specific n, one can calculate this bound, and it is much smaller than the general bounds above. For example, we have $|t_{60}| \le 5.96 \cdot 10^{-14}$ with this method, whereas Lemma A.1 would give the bound $1.58 \cdot 10^6$.

Lemma A.3. *Let $N \in \mathbb{N}$ and $x \in \mathbb{C}$, and let c and $g(n)$ be as in Lemma A.1. Set*

$$q = \frac{e^{g(N)}\,|x|}{e^c\sqrt{N^{b-1}}}$$

and suppose $q < 1$ holds. We get the bound

$$\left|\sum_{n\ge N} t_n x^n\right| \le \frac{q^N}{1-q}$$

for the tails of the infinite sum in the determinant T. For the tails of the determinant S we get an additional factor $b! + 1$ in the bound.

Proof. By Lemma A.1 we have

$$|t_n| \le \exp\left(-\frac{b-1}{2}n\log n - cn + n\,g(n)\right).$$

Using monotonicity yields

$$\left|\sum_{n\ge N} t_n x^n\right| \le \sum_{n\ge N}\left(\frac{e^{g(n)}\,|x|}{e^c\sqrt{n^{b-1}}}\right)^n \le \sum_{n\ge N}\left(\frac{e^{g(N)}\,|x|}{e^c\sqrt{N^{b-1}}}\right)^n = q^N\frac{1}{1-q}.$$

□

Remark A.4. We can also get tighter bounds in Lemma A.3 using the ideas presented in Remark A.2. We can even use combinations of those bounds: For $M > N$, we separate

$$\left|\sum_{n\geq N} t_n x^n\right| \leq \sum_{M>n\geq N} |t_n|\,|x|^n + \left|\sum_{n\geq M} t_n x^n\right|$$

and use the bound (A.1) for $M > n \geq N$ and Lemma A.3 (tightened by some ideas from Remark A.2) for the sum over $n \geq M$. For example, we obtain the tail-bound

$$\left|\sum_{n\geq 60} t_n x^n\right| \leq 8.051 \cdot 10^{-14} + 3.627 \cdot 10^{-14}$$

for $|x| \leq 1$, where $M = 86$ was chosen. (We will denote the constant on the right hand side of the inequality above by $B_{T_{60}}$, see the proof of Lemma C.1.) Using Lemma A.3 directly would just give 0.0041.

Appendix B. Supplement to Section 3

Proof of Lemma 3.1. First of all, we rule out the possibility that $Q(x)$ is entire by providing a lower bound for the coefficients $q_b(m)$. To this end, consider compositions of 1 consisting of $b-1$ copies of $b^{-1}, b^{-2}, \ldots, b^{1-m}$ and b copies of b^{-m}. Since there are $\frac{((b-1)m+1)!}{((b-1)!)^{m-1}b!}$ possible ways to arrange them in an order, we know that

$$q_b(m) \geq \frac{((b-1)m+1)!}{((b-1)!)^{m-1}b!},$$

from which it follows that the radius of convergence of $Q(x)$ is at most $(b-1)!$. Since all coefficients are positive, Pringsheim's theorem guarantees that the radius of convergence, which we denote by ρ_b, is also a singularity. We already know that $Q(x)$ is meromorphic (being the quotient of two entire functions), hence ρ_b is a pole singularity. Let p be the pole order, and set $w_r = \lim_{x\to\rho_b^-}(x-\rho_b)^p\, Q_r(x)$, which must be nonnegative and real. Multiplying the matrix equation (2.3) by $(x-\rho_b)^p$ and taking the limit, we see that $\mathbf{w} = (w_1, w_2, \ldots)^T$ is a right eigenvector of $\mathbf{M}(\rho_b)$. Since all entries in $\mathbf{M}(\rho_b)$ are nonnegative and those on and above the main diagonal are strictly positive, it follows that $w_r > 0$ for all r, i.e., all functions $Q_r(x)$ have the same pole order (as $Q(x)$).

Now we split the identity (2.3). Let $m_{11} = x/(b-1)!$ be the first entry of $\mathbf{M}(x)$, $\mathbf{c}$ the rest of the first column, $\mathbf{r}$ the rest of the first row and $\overline{\mathbf{M}}$ the matrix obtained from $\mathbf{M}$ by removing the first row and the first column. Moreover, $\overline{\mathbf{V}}$ is obtained from $\mathbf{V}$ by removing the first entry $Q_1(x)$. Now we have

$$Q_1(x) = m_{11}\, Q_1(x) + \mathbf{r}\,\overline{\mathbf{V}} + \frac{x}{b!} \tag{B.1}$$

and

$$\overline{\mathbf{V}} = \mathbf{c}\, Q_1(x) + \overline{\mathbf{M}}\,\overline{\mathbf{V}},$$

from which we obtain

$$\overline{\mathbf{V}} = (\mathbf{I} - \overline{\mathbf{M}})^{-1}\mathbf{c}\, Q_1(x)\,. \tag{B.2}$$

Once again, the inverse $(\mathbf{I} - \overline{\mathbf{M}})^{-1}$ exists a priori in the ring of formal power series, but one can show that $\det(\mathbf{I} - \overline{\mathbf{M}})$ is in fact an entire function, so the entries of the inverse are all meromorphic (see again the calculations in Appendix A). Moreover, $(\mathbf{I} - \overline{\mathbf{M}})^{-1}\mathbf{c}$ cannot have a singularity at ρ_b or at any smaller positive real number, because if this was the case, the right hand side of (B.2) would have a higher pole order at that point than the left hand side. Since it has positive coefficients only (the inverse can be expanded into a geometric series), its entries must be analytic in a circle of radius $> \rho_b$ around 0. Now we substitute (B.2) in (B.1) to obtain

$$Q_1(x) = m_{11}\, Q_1(x) + \mathbf{r}(\mathbf{I} - \overline{\mathbf{M}})^{-1}\mathbf{c}\, Q_1(x) + \frac{x}{b!}$$

and thus

$$Q_1(x) = \frac{x}{b!}\left(1 - m_{11} - \mathbf{r}(\mathbf{I} - \overline{\mathbf{M}})^{-1}\mathbf{c}\right)^{-1}.$$

Note that

$$R(x) = m_{11} + \mathbf{r}(\mathbf{I} - \overline{\mathbf{M}})^{-1}\mathbf{c}$$

has only positive coefficients, so $R(x) = 1$ has a unique positive real solution, which must be ρ_b (recalling that $R(x)$ is analytic in a circle of radius $> \rho_b$ around 0). Of course, $R'(\rho_b) > 0$, so its multiplicity is 1, which means that ρ_b is a simple pole. Moreover, by the triangle inequality there are no complex solutions of $R(x) = 1$ with the same modulus, which means that there are no further singularities of $Q_1(x)$ (and thus $Q(x)$) in a circle of radius $\rho_b + \epsilon_b$ around 0 for suitable $\epsilon_b > 0$. □

Appendix C. Supplement to Section 4

Lemma C.1. *The function $T(x)$ has exactly one zero with $|x| < \frac{3}{2}$. This simple zero lies at $x_0 = 0.83845184342\ldots$.*

Remark C.2. Note that $1/x_0 = \gamma = 1.192674341213\ldots$.

Proof. We have $|T(x) - T_{60}(x)| \le B_{T_{60}}$ with $B_{T_{60}} = 1.17 \cdot 10^{-13}$, see Lemma A.3 and Remark A.4. On the other hand, we have $|T_{60}(x)| > 0.062$ for $|x| = \frac{3}{2}$ (the minimum is attained on the positive real axis). Therefore, the functions $T(x)$ and $T_{60}(x)$ have the same number of zeros inside a disk $|x| < \frac{3}{2}$ by Rouché's theorem ($0.062 > B_{T_{60}}$). This number equals one, since there is only one zero, a simple zero, of $T_{60}(x)$ with absolute value smaller than $\frac{3}{2}$.

To find the exact position of that zero consider $T_{60}(x) + B_{T_{60}} I$ with the interval $I = [-1, 1]$. Using a bisection method (starting with $\frac{3}{2}I$) together with interval arithmetic, we find an interval that contains x_0. From this, we can extract correct digits of x_0. □

Appendix D. Supplement to Section 5

In this section of the appendix we gather some properties of the solutions $x = \theta(u)$ of the functional equation $W(x) = 1/u$.

Lemma D.1. *For $u \in \mathbb{C}$ with $|u| \le 1$, each root x of $W(x) = 1/u$ fulfils $|x| \ge \theta$, where equality holds only if x is real and positive.*

Proof. Let $u \in \mathbb{C}$ with $|u| \le 1$. By the nonnegativity of the coefficients of W and by using the triangle inequality, we have

$$W(\theta) = 1 \le |1/u| = |W(x)| \le W(|x|).$$

The first part of the lemma follows, since W is increasing on the positive real line.

Since the coefficients of W are indeed positive, the power series W is aperiodic (i.e., the *gcd* of all exponents whose associated coefficients are not zero is 1), and therefore, the inequality $|W(x)| \le W(|x|)$ is strict, i.e., we have $W(\theta) < W(|x|)$, unless x is real and positive. Again, monotonicity finishes the proof. □

The following lemma tells us that the dominant root of $W(x) = 1$ is the simple zero θ.

Lemma D.2. *There exists exactly one root of $W(x) = 1$ with $|x| \le \theta$, namely θ. Further, θ is a simple root, and there exists an $\epsilon > 0$ such that θ is the only root of $W(x) = 1$ with absolute value less than $\theta + \epsilon$.*

Proof. By Lemma D.1 with $u = 1$, the positive real θ is the unique root of $W(x) = 1$ with minimal absolute value. This proves the first part of the lemma.

Using Theorem III, we get

$$|W(x)| \le O\left(\sum_{n=1}^{\infty} (\gamma\,|x|)^n\right),$$

which is bounded for $|x| < 1/\gamma$. Therefore, the radius of convergence r of W is at least $1/\gamma > \theta$, and W is holomorphic and thus analytic for $|x| < r$. Since zeros of analytic functions do not accumulate, the existence of an $\epsilon > 0$ as desired follows.

The root θ is simple, since W is strictly increasing on the interval $(0, r)$ (all coefficients of W are positive). □

Lemma D.3. *For $u \in \mathbb{C}$ let $|\theta(u)| = |x|$, where x is a root of $W(x) = 1/u$ with smallest absolute value. Then, for $\varphi \in [-\pi, \pi]$, the function $\left|\theta(e^{i\varphi})\right|$ attains a unique minimum at $\varphi = 0$.*

Proof. Set $u = e^{i\varphi}$ and use Lemma D.1: we obtain $|\theta(u)| \ge \theta$. If $\varphi \neq 0$, then $\theta(u)$ is not a positive real number, therefore we have $|\theta(u)| > \theta$. □

Proof of Theorem V. Set

$$g_n(u) = \sum_{s=1}^{\infty} \frac{\mathcal{W}_b(s,n)}{n!} u^s.$$

We extract g_n from the bivariate generating function $G(x, u)$. In order to do so, we proceed as in Theorem IX.9 (singularity perturbation for meromorphic functions) of Flajolet and Sedgewick [8]. First, we check that all requirements are fulfilled.

By Lemma D.2, the function $G(x, 1)$ has a dominant simple pole at $x = \theta$ and no other singularities with absolute values smaller than $\theta + \epsilon$. The denominator $1 - u\,W(x)$ is analytic and not degenerated at $(x, u) = (\theta, 1)$; the latter since its derivative with respect to x is $W'(\theta) \neq 0$ (θ is a simple root of F) and its derivative with respect to u is $-\,W(\theta) = -1 \neq 0$. Thus, there exists a nonconstant function $\theta(u)$ with the following properties: it is analytic at $u = 1$, it fulfils $\theta(1) = \theta$, and we have $W(\theta(u)) = 1/u$.

Therefore, by Cauchy's integral formula and the residue theorem, we obtain

$$\begin{aligned} g_n(u) &= -\operatorname{Res}\left(\frac{1}{1 - u\,W(x)} x^{-n-1}, x = \theta(u)\right) + \frac{1}{2\pi i}\oint_{|x|=\theta+\epsilon} G(x,u)\,\frac{\mathrm{d}z}{z^{n+1}} \\ &= \frac{1}{u\,\theta(u)\,W'(\theta(u))}\left(\frac{1}{\theta(u)}\right)^n + O\big((\theta+\epsilon)^{-n}\big) \end{aligned}$$

for u in a suitable neighbourhood of 1.

Next, use the local version of the quasi-power theorem, see Theorem IX.14 of [8] or Hwang's original paper [10]. Set

$$A(u) = \big(u\,\theta(u)\,W'(\theta(u))\big)^{-1}$$

and

$$B(u) = (\theta(u))^{-1},$$

so that $g_n(u) = A(u)B(u)^n + O((\theta+\epsilon)^{-n})$. Again, we have to check some requirements. Since $\theta(u) \neq 0$ for u in a suitable neighbourhood of 0, the function B is analytic at zero, and so is the function A (since W is analytic in a neighbourhood of $\theta(1) = \theta$ as well and has nonzero derivative there). Moreover, we use the fact that $\left|\theta\left(e^{i\varphi}\right)\right|$ has a unique minimum at $\varphi = 0$ (cf. Lemma D.3).

As a result, Theorem IX.14 of [8] gives us

$$\begin{aligned} \frac{\mathcal{W}_b(s,n)}{n!} &= \frac{A(1)\,B(1)^n}{\sigma\sqrt{2\pi n}} \exp\left(-\frac{z^2}{2\sigma^2}\right)\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) \\ &= \frac{\nu^n}{\theta\,W'(\theta)\,\sigma\sqrt{2\pi n}} \exp\left(-\frac{z^2}{2\sigma^2}\right)\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right), \end{aligned} \tag{D.1}$$

where $z = (s - \mu n)/\sqrt{n}$. Mean and variance can be calculated as follows. We have

$$\mu = \frac{B'(1)}{B(1)} = -\frac{\theta'(1)}{\theta(1)} = \frac{1}{\theta\,W'(\theta)},$$

and $\sigma > 0$ is determined by

$$\begin{aligned} \sigma^2 &= \frac{B''(1)}{B(1)} + \frac{B'(1)}{B(1)} - \left(\frac{B'(1)}{B(1)}\right)^2 = -\frac{\theta''(1)}{\theta(1)} - \frac{\theta'(1)}{\theta(1)} + \left(\frac{\theta'(1)}{\theta(1)}\right)^2 \\ &= \frac{W''(\theta)}{\theta\,W'(\theta)^3} - \frac{1}{\theta\,W'(\theta)} + \frac{1}{\theta^2\,W'(\theta)^2}, \end{aligned}$$

where we used implicit differentiation of $W(\theta(u)) = 1/u$ to get expressions for $\theta'(u)$ and $\theta''(u)$.

When $s = \mu n + O(1)$, the value $\frac{\mathcal{W}_b(s,n)}{n!}$ is maximal with respect to s. Its value can then be calculated by (D.1). ☐

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