ANALYSIS OF DIGITAL EXPANSIONS IN CONJUNCTION WITH KOBLITZ CURVES IN CHARACTERISTIC THREE

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Consider the elliptic curve

$$\mathcal{E}_3 : Y^2 = X^3 - X - 1$$

defined over $\mathbb{F}_3$. This curve was studied by Koblitz [3]. In cryptographic applications we are interested in the group $\mathcal{E}_3(\mathbb{F}_{3^m})$ of rational points over a field extension $\mathbb{F}_{3^m}$ of $\mathbb{F}_3$ for an $m \in \mathbb{N}$. The Frobenius endomorphism

$$\varphi : \mathcal{E}_3(\mathbb{F}_{3^m}) \longrightarrow \mathcal{E}_3(\mathbb{F}_{3^m}), \quad (x, y) \longmapsto \left(x^3, y^3\right)$$

satisfies the relation $\varphi^2 - 3\varphi + 3 = 0$, so φ may be identified with the imaginary quadratic number $\tau = \frac{3}{2} + \frac{1}{2}\sqrt{-3}$, which is a solution of the mentioned relation. Thus we have an isomorphism between $\mathbb{Z}[\tau]$ and the endomorphism ring of $\mathcal{E}_3(\mathbb{F}_{3^m})$.

Let $z \in \mathbb{Z}[\tau]$ and $P \in \mathcal{E}_3(\mathbb{F}_{3^m})$. If we write the element z as $\sum_{j=0}^{\ell-1} z_j \tau^j$ for some digits z_j belonging to a digits set $\mathcal{D}$, then we can compute the action zP as $\sum_{j=0}^{\ell-1} z_j \, \varphi^j(P)$ via a Horner scheme. The resulting *Frobenius-and-add* method [2, 4, 5] is much faster than the classic double-and-add scalar multiplication.

So we are interested in a τ-adic expansion for an element of $\mathbb{Z}[\tau]$, such that the operation mentioned above is as efficient as possible. Let $w \in \mathbb{N}$ with $w \geq 2$ and suppose our digit set $\mathcal{D}$ consists of zero and all minimal norm representatives modulo τ^w not divisible by τ, cf. Solinas [4, 5]. We consider numbers $z = \sum_{j=0}^{\ell-1} z_j \tau^j$, where $z_j \in \mathcal{D}$ and $z_{\ell-1} \ldots z_0$ is a *width-w non-adjacent form*, or *w-NAF* for short, i.e. each block of length w contains at most one non-zero. Figure 1 shows two examples. There all w-NAFs of a given length ℓ are drawn.

Since a w-NAF can be described by the regular expression

$$\left(\varepsilon + \sum_{d \in \mathcal{D}\setminus\{0\}} \sum_{k=0}^{w-2} 0^k d\right) \left(0 + \sum_{d \in \mathcal{D}\setminus\{0\}} 0^{w-1} d\right)^*,$$

we can simply use a generating function to count the occurrences of a non-zero digit in all w-NAFs of length ℓ. We obtain

$$\frac{1}{(\mathcal{N}(\tau)-1)w+1} \, \mathcal{N}(\tau)^{\ell+w} + \mathcal{O}\Big(\big(\rho\,\mathcal{N}(\tau)\big)^{\ell}\Big),$$

where $\rho = (1 + \frac{1}{\mathcal{N}(\tau)w^3})^{-1} < 1$ and $\mathcal{N}(\tau)$ denotes the norm of τ. This result holds for a general algebraic integer τ, too.

A more general question is, what the number of occurrences of a non-zero digit η is, when we look at all w-NAFs with absolute value smaller than a given N. For imaginary quadratic τ with $|\tau| > 1$, the answer is given by the following theorem.

Key words and phrases. Koblitz curves, Frobenius endomorphism, tau-adic expansions, non-adjacent forms, sum of digits.

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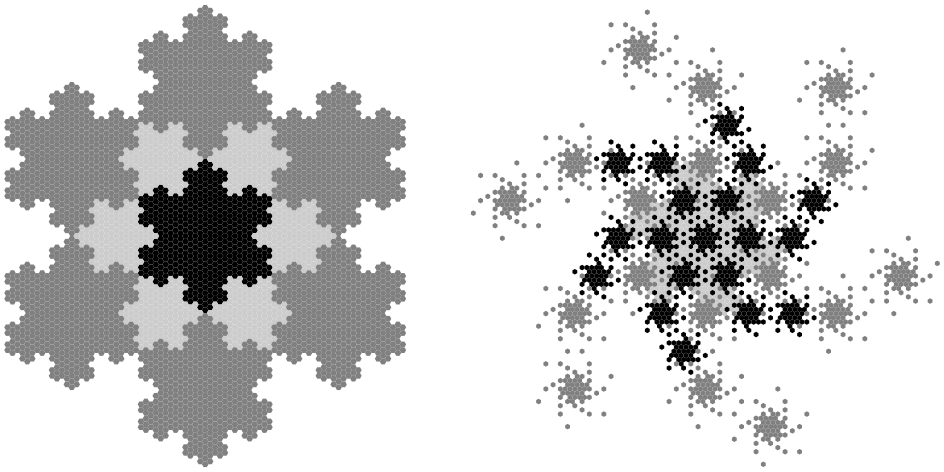

(a) $\tau = \frac{3}{2} + \frac{1}{2}\sqrt{-3}$, $w = 2$ and $\ell = 7$ (b) $\tau = \frac{3}{2} + \frac{1}{2}\sqrt{-3}$, $w = 3$ and $\ell = 6$

Figure 1: Values of w-NAFs of length ℓ.

Theorem. *Let $0 \neq \eta \in \mathcal{D}$, $U \subseteq \mathbb{C}$ be the unit disc and let $N \in \mathbb{R}_{\geq 0}$. Then the number of occurrences of the digit η in all w-NAFs with value in the region NU is*

$$Z_{\tau,w,\eta}(N) = e_w N^2 \, \lambda(U) \log_{|\tau|} N + N^2 \, \psi_\eta\Big(\log_{|\tau|} N\Big) + \mathcal{O}(N^\alpha)\,.$$

There $e_w = \Big(\mathcal{N}(\tau)^{w-1}\left((\mathcal{N}(\tau)-1)w+1\right)\Big)^{-1}$, the function ψ_η is 1-periodic and continuous, and α is a computable constant with $\alpha < 2$.

Instead of the unit disc, other bounded regions U can be used, but they possibly destroy the periodicity and continuity of ψ_η. The proof of the theorem follows the ideas of Delange [1].

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