The Number of Multi-Base Representations of an Integer

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(joint work with Dimbinaina Ralaivaosaona and Stephan Wagner)

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A Hint . . .

Multi-Base Representations

Representations

$$n = \sum_j d_j p_1^{\alpha_{1j}} p_2^{\alpha_{2j}} \dots p_m^{\alpha_{mj}}$$

- digits d_j out of digit set $\{0, 1, \dots, d-1\}$
- bases $p_1, \dots, p_m$ (coprime positive integers)
- non-negative integers α_{ij}
- all $p_1^{\alpha_{1j}} p_2^{\alpha_{2j}} \dots p_m^{\alpha_{mj}}$ distinct

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Question

How many representations does a number have?

2–3-Expansions

- set-up
  - bases 2 and 3
  - digits 0 and 1
- partitions into powers of 3
  - expansion of n

$$n = b_0 + 3b_1 + 9b_2 + \cdots + 3^\ell b_\ell$$

  - b_j in binary

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Recursion

number of representations

$$P_n = \begin{cases} P_{n-1} + P_{n/3} & \text{if } 3 \mid n \\ P_{n-1} & \text{if } 3 \nmid n \end{cases}$$

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- $\rightsquigarrow$ asymptotic formula
- generalization to 2–p-expansions

Number of Representations

Theorem (K–Ralaivaosaona–Wagner 2014)

- *fix m bases, fix digit set* $\{0, \dots, d-1\}$
- *number of multi-base representations* P_n
  - *if* $m \geq 3$

$$\log P_n = \kappa (\log n)^m + C_1 (\log n)^{m-1} \log\log n + C_2 (\log n)^{m-1} + O\big((\log n)^{m-2} \log\log n\big)$$

  - *if* $m = 2$

$$P_n = K(n) (\log n)^{K_1} n^{K_2} \exp\Big(\kappa \log^2\Big(\frac{n}{\log n}\Big)\Big)$$

  - *with*

$$\kappa = \frac{\log d}{m!} \prod_{i=1}^{m} \frac{1}{\log p_i}$$

The Generating Function

- representations

$$n = \sum_j d_j p_1^{\alpha_{1j}} p_2^{\alpha_{2j}} \dots p_m^{\alpha_{mj}}$$

  - digits $d_j \in \{0, 1, \dots, d-1\}$
  - power products $\mathcal{B} = \{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m} \mid \alpha_i \in \mathbb{N}_0\}$

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Generating Function

$$F(z) = \sum_{n \in \mathbb{N}_0} P_n z^n = \prod_{b \in \mathcal{B}} \left(1 + z^b + z^{2b} + \dots + z^{(d-1)b}\right)$$

?

Saddle-Point Method

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- extract coefficients (Cauchy's integral formula)

$$P_n = [z^n]F(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} F(z) \frac{dz}{z^{n+1}}$$

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$$P_n = [z^n]F(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} F(z) \frac{dz}{z^{n+1}}$$

- substitute $z = e^{-(r+i\tau)}$

$$P_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(nr + f(r+i\tau) + in\tau\right) d\tau$$

with $f(r+i\tau) = \log F(e^{-(r+i\tau)})$

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- asymptotics

$$P_n \sim \frac{e^{nr+f(r)}}{2\pi} \int_{-\infty}^{\infty} \exp\left(-f''(r)\frac{\tau^2}{2}\right) d\tau = \frac{e^{nr+f(r)}}{\sqrt{2\pi f''(r)}}$$

Mellin & Friends

- function

$$f(r) = \sum_{b \in \mathcal{B}} \log\left(1 + e^{-br} + e^{-2br} + \dots + e^{-(d-1)br}\right)$$

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$$Y(s) = \int_0^\infty \log\left(1+e^{-r}+e^{-2r}+\cdots+e^{-(d-1)r}\right) r^{s-1}\, dr \underset{s\to 0}{\sim} \frac{\log d}{s}$$

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- Dirichlet series

$$D(s) = \sum_{b \in \mathcal{B}} b^{-s} = \prod_{i=1}^{m} \frac{1}{1 - p_i^{-s}} \underset{s\to 0}{\sim} \prod_{i=1}^{m} \frac{1}{s \log p_i}$$

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- inverse Mellin transform

$$f(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Y(s)D(s)r^{-s}\, ds \underset{r\to 0^+}{\sim} \frac{a_m}{m!}(\log 1/r)^m$$

Tails

- step one

$$\frac{F(z)}{F(|z|)} \leq \exp\left(-C \sum_{b \in \mathcal{B}(r)} \|by\|^2\right)$$

1

  - with $z = e^{-r+2\pi iy}$
  - $\mathcal{B}(r)$ for $\mathcal{B} \cap [1, 1/r] = \{b \in \mathcal{B} \mid br \leq 1\}$
  - distance to nearest integer $\|\cdot\|$

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2

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$$\sum_{b \in \mathcal{B}(r)} \|by\|^2 \geq (\log(1/r))^{m-1} \times \text{something}$$

  - Dirichlet's approximation theorem
  - pigeonhole principle
  - problem if $m = 2$: valid except y in “small” set

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- step three
  - apply bounds

Plugging Everything Together...

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Also Possible

- sum of digits
- Hamming weight